THE KDV CURVE AND SCHRÖDINGER-AIRY CURVE

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Abstract. Among other things, we introduce the notion of KdV curves and Schrödinger-Airy curves. These curves are stable solutions to the geometric KdV-Airy flow equation and Schrödinger-Airy flow equation respectively, which were recently proposed by Sun and Wang. We demonstrate that the KdV curves can be regarded as a 3rd-order analogue of geodesics. Other interesting properties of these curves will be addressed. Explicit examples of these curves will be provided. In addition, we will consider a perturbed KdV curve system and show the existence of multiple solutions to this system on the torus.

1. The KdV curve

Suppose that \((N, \omega, J)\) is a Kähler manifold with symplectic form \(\omega\) and complex structure \(J\) and that \(u(x, t)\) is a smooth map from \(S^1 \times \mathbb{R}\) into \(N\). Let \(\nabla_x\) denote the covariant derivative \(\nabla_\frac{\partial}{\partial x}\) on the pull-back bundle \(u^{-1}TN\) induced from the Levi-Civita connection \(\nabla\) on \(N\). In [5], Sun and Wang introduced the so-called geometric KdV-Airy flow as follows:

\begin{equation}
\frac{\partial u}{\partial t} = \nabla^2_x u_x + \frac{1}{2} R(u_x, J u_x) J u_x,
\end{equation}

where \(R\) is the curvature tensor on \(N\). Here we denote \(\frac{\partial u}{\partial t} = \nabla_t u, u_x = \nabla_x u\) and write \(J = J(u)\) for simplicity. Equation (1.1) is a geometric flow which stems from the vortex filament dynamics and belongs to the same family as the Schrödinger flow (cf. [4]):

\begin{equation}
\frac{\partial u}{\partial t} = J \nabla_x u_x.
\end{equation}

In certain circumstances, equation (1.1) transforms into the well-known modified KdV equation. We refer to [5] for more details and background knowledge.

The KdV curve is defined as the stable solution to equation (1.1). Namely, a map \(u \in C^\infty(S^1, N)\) is called a KdV curve if it satisfies

\begin{equation}
\nabla^2_x u_x + \frac{1}{2} R(u_x, J u_x) J u_x = 0.
\end{equation}

Suppose \(N\) is embedded in a Euclidean space \(\mathbb{R}^K\). Define the KdV energy functional on Sobolev space,

\[ W = W^{2,2}(S^1, N) := \{W^{2,2}(S^1, \mathbb{R}^K) | u(x) \in N \text{ for a.e. } x \in S^1\}, \]
as follows:

\begin{equation}
F(u) = \frac{1}{2} \int_{S^1} \langle \nabla_x u_x, J u_x \rangle dx.
\end{equation}

Let \( u_s : [0, \delta] \to W \) be a variation of \( u \) and

\[
\frac{\partial u_s}{\partial s} \bigg|_{s=0} = \xi \in T_u W = W^{2,2}(S^1, T_u N).
\]

Then a direct computation yields

\[
(dF(u), \xi) = \frac{d}{ds} F(u_s) \bigg|_{s=0} = \int_{S^1} \langle J \nabla_x^2 u_x + \frac{1}{2} R(J u_x, u_x)u_x, \xi \rangle.
\]

So the KdV curves are actually critical points of the functional \( F \).

There is a symplectic form \( \Omega \) on \( W \) naturally induced by the symplectic form \( \omega \) on \( N \). Indeed, for any vector fields \( X, Y \in T_u W \), we can define

\[
\Omega(X, Y) = \int_{S^1} \omega(X(u)), Y(u))dx.
\]

Similarly, there is an induced complex structure on \( W \), which we still denote by \( J \).

The Hamiltonian vector field \( X_F \) associated with \( F \) is defined by

\[
\Omega(X_F, \cdot) = dF.
\]

Then the KdV-Airy flow (1.1) can be written in the following form:

\[
\begin{aligned}
\frac{du_t}{dt} &= X_F = J \nabla F(u), \\
\end{aligned}
\]

which is the Hamiltonian flow of \( F \) on the infinite dimensional symplectic manifold \( (W, \Omega) \). On the other hand, the Schrödinger flow (1.2) is known to be the Hamiltonian flow of the normal energy functional

\[
E(u) = \frac{1}{2} \int_{S^1} |u_x|^2 dx,
\]

and the stable solutions for Schrödinger flow are just geodesics. It turns out that the KdV-Airy flow and the Schrödinger flow belong to the same integrable system [5]. Therefore, the KdV curves are not only important for understanding the KdV-Airy flow, but are also of special interest as a higher order analogue of geodesics.

It is a basic fact that geodesics have constant speed. In other words, the energy density \( e(u) = |u_x|^2 \) remains the same along geodesics. There is an analogous result for KdV curves.

**Theorem 1.1.** For any given KdV curve \( u \), the quantity

\[
f(u) = \langle \nabla_x u_x, J u_x \rangle
\]

is a constant.

**Proof.** A simple calculation yields

\[
\begin{aligned}
d_x f(u) &= d_x \langle \nabla_x u_x, J u_x \rangle \\
&= \langle \nabla_x^2 u_x, J u_x \rangle + \langle \nabla_x u_x, J \nabla_x u_x \rangle \\
&= -\frac{1}{2} \langle R(u_x, J u_x)u_x, J u_x \rangle \\
&= 0.
\end{aligned}
\]

In fact, on some manifolds, KdV curves are just geodesics.
Theorem 1.2. If the holomorphic sectional curvature $k$ of $N$ is non-positive, the KdV curves are geodesics. In particular, if $k$ is strictly negative, then the KdV curves are just constant maps.

Proof. On manifolds with non-positive holomorphic sectional curvature, we have

$$k(X) = \frac{R(X, JX, JX, X)}{|X|^4} \leq -\delta, \quad \delta \geq 0.$$ 

For any KdV curve $u$ which satisfies equation (1.3), multiplying by $u_x$ and integrating by parts, we have

$$0 = \int \langle \nabla^2 u_x, u_x \rangle + \frac{1}{2} \int R(u_x, Ju_x, Ju_x, u_x)$$
$$= -\int |\nabla_x u_x|^2 + \frac{1}{2} \int k(u_x)|u_x|^4$$
$$\leq -\int |\nabla_x u_x|^2 - \frac{1}{2}\delta \int |u_x|^4.$$ 

This implies that $\nabla_x u_x = 0$, which means that $u$ is a geodesic. Moreover, if $\delta > 0$, then $u_x \equiv 0$, which means that $u$ is a constant map. 

2. Schrödinger-Airy curve

In [5], the authors also addressed the so-called Schrödinger-Airy flow, which can be regarded as a geometric generalization of the Hirota equation. Explicitly, the flow is defined as follows:

$$u_t = \alpha J\nabla_x u_x + \beta (\nabla^2_x u_x + \frac{1}{2} R(u_x, Ju_x)Ju_x),$$

where $\alpha$ and $\beta$ are two positive numbers. We call a stable solution to equation (2.1) a Schrödinger-Airy curve. Namely, a Schrödinger-Airy curve is a map $u \in C^\infty(S^1, N)$ which satisfies the equation

$$\nabla_x^2 u_x + \frac{1}{2} R(u_x, Ju_x)Ju_x = \lambda J\nabla_x u_x,$$

where $\lambda$ is a function on $S^1$. It’s a natural extension of the KdV curve. Particularly, a KdV curve satisfies equation (2.2) for $\lambda \equiv 0$. If we regard $\lambda$ as a Lagrange multiplier, then the Schrödinger-Airy curve is a critical point of $F(u)$ under the constraint $E(u) = \text{const.}$

Note that both the KdV curve and the Schrödinger-Airy curve can be defined weakly in $W^{2,2}(S^1, N)$. In fact, we may assume that the compact manifold $N$ is embedded in a Euclidean space $\mathbb{R}^K$. Denote the inner product on $\mathbb{R}^K$ by $(\cdot, \cdot)$ and the second fundamental form of $N$ by $A$. Then we have

$$\nabla_x u_x = u_{xx} - A(u)(u_x, u_x).$$

It follows that

$$\nabla^2_x u_x = (\nabla_x u_x)_x - A(u)(u_x, \nabla_x u_x)$$
$$= u_{xxx} - [A(u)(u_x, u_x)]_x - A(u)(u_x, \nabla_x u_x)$$
$$= u_{xxx} - \nabla A(u)(u_x, u_x, u_x) - 3A(u)(u_x, \nabla_x u_x).$$

(2.3)
Therefore, we may call a map $u \in W^{2,2} (S^1, N)$ a weak KdV curve if $u$ is a $W^{2,2}$-weak solution of equation (1.3). Namely, $u$ is a weak KdV curve if for any $v \in C^\infty (S^1, \mathbb{R}^K)$, it follows that

$$\int_{S^1} (\nabla_x u_x, v_x) dx + \int_{S^1} (A(u) (u_x, \nabla_x u_x), v) dx - \frac{1}{2} \int_{S^1} (R(u_x, Ju_x) Ju_x, v) = 0.$$ 

Similarly a weak Schrödinger-Airy curve is defined as a $W^{2,2}$-weak solution to equation (2.2). The following theorem shows that a weak Schrödinger-Airy (or KdV) curve is actually smooth.

**Theorem 2.1.** A $W^{2,2}$-weak Schrödinger-Airy curve is smooth.

**Proof.** By (2.5), equation (2.2) can be rewritten as

$$u_{xxxx} = f(u_x, u_{xx}),$$

where

$$f(u_x, u_{xx}) = \nabla A(u)(u_x, u_x) + 3A(u)(u_x, \nabla_x u_x) - \frac{1}{2} R(u_x, Ju_x) Ju_x + \lambda J \nabla_x u_x.$$ 

It is obvious that equation (2.4) is an elliptic equation for $u_x$, and $u_x$ is a $W^{1,2}$-weak solution to (2.4) if $u$ is a $W^{2,2}$-weak solution to (2.2). By Sobolev embedding theorems, $u \in W^{2,2}(S^1, N)$ implies $u_x \in C^\alpha$ for some $\alpha \in (0, 1)$. Hence $f(u_x, u_{xx}) \in L^2$. Now by the standard $L^2$-estimate, it follows that $u_x \in W^{2,2}$ from equation (2.4). A bootstrapping argument then proves the theorem. □

In the remaining part of this section, we always suppose that $N$ is a closed Riemann surface. In this case, we shall show that Schrödinger-Airy curves satisfy some nice properties. In particular, all these properties hold for KdV curves.

First we observe that the functional $F$ is not a geometric invariant. That is, it depends on the choice of the parameter of the curve $u$. Let $s = \int_0^s |u_x| dx$ be the arc length parameter of $u$. Let $t = u_s$ be the unit tangent vector and $n$ be the unit normal vector orthogonal to $t$. We denote $\nabla_s = \nabla_{u_s}$. Then

$$\nabla_s u_s = k_g n,$$

where $k_g$ is the geodesic curvature. It follows that

$$\langle \nabla_s u_s, Ju_s \rangle = \langle k_g n, J t \rangle = k_g.$$ 

Note that $k_g$ in the last equality may vary from the usual definition of geodesic curvature by a minus sign, depending on the complex structure $J$. But here we only care about the absolute value of $k_g$ and ignore the slight difference of sign. On the other hand, $ds = |u_x| dx$. Therefore, we get

$$\langle \nabla_x u_x, Ju_x \rangle = |u_x|^3 \langle \nabla_s u_s, Ju_s \rangle + |u_x|^3 |\nabla_s u_x| \langle u_s, Ju_s \rangle = |u_x|^3 k_g.$$ 

**Theorem 2.2.** Suppose $N$ is a Riemann surface and $u$ is a Schrödinger-Airy curve. If $u$ has constant speed $|u_x| = c$, then $u$ has constant geodesic curvature.

**Proof.** Multiplying equation (2.2) by $Ju_x$, we get

$$\langle \nabla_x^2 u_x, Ju_x \rangle = -\lambda \langle \nabla_x u_x, u_x \rangle.$$ 

By (2.5), the geodesic curvature $k_g$ satisfies

$$\langle \nabla_x u_x, Ju_x \rangle = |u_x|^3 k_g.$$
Note that
\[ \nabla_x \langle \nabla_x u_x, J u_x \rangle = \langle \nabla_x^2 u_x, J u_x \rangle. \]
Hence equality (2.6) is equivalent to
\[ \nabla_x (|u_x|^3 k_g) = -\frac{\lambda}{2} \nabla_x |u_x|^2. \]
It is easy to see that if \(|u_x| = c\) is a constant, then \(\nabla_x k_g = 0\), as desired.

Next we suppose in addition that \(N\) has constant curvature. We show that in this case the geodesic curvature can be computed.

**Theorem 2.3.** Suppose \(N\) is a Riemann surface with constant curvature \(k\) and \(u\) is a non-trivial Schrödinger-Airy curve. Assume \(u\) has constant speed \(|u_x| = c\).

Then the geodesic curvature \(k_g\) is determined by \(\lambda\).

**Proof.** For a constant curvature surface \(N\), we have
\[ (2.7) \quad R(u_x, J u_x) J u_x = k |u_x|^2 u_x. \]
Since \(u\) has constant speed, we have
\[ 0 = \nabla_x |u_x|^2 = \langle \nabla_x u_x, u_x \rangle. \]
Thus we may suppose \(\nabla_x u_x = \alpha J u_x\) for some function \(\alpha(x)\). It follows from (2.5) that
\[ \langle \nabla_x u_x, J u_x \rangle = -|u_x|^2 \alpha = |u_x|^3 k_g. \]
Hence \(\alpha = -ck_g\), which is constant by Theorem 2.2. Consequently, we have
\[ \nabla_x u_x = -ck_g J u_x, \quad \nabla_x^2 u_x = -c^2 k_g^2 u_x. \]
Then using equations (2.2) and (2.7), we get
\[ \nabla_x^2 u_x + \frac{1}{2} R(u_x, J u_x) J u_x = -c^2 k_g^2 u_x + \frac{1}{2} c^2 ku_x = \lambda J \nabla_x u_x = \lambda ck_g u_x. \]
This simplifies to
\[ k_g^2 + \frac{\lambda}{c} k_g - \frac{1}{2} k = 0. \]
This is a second order equation for \(k_g\), and the solution is given by
\[ (2.8) \quad k_g = \frac{\lambda \pm c \sqrt{\lambda^2 + 2kc^2}}{2c}, \]
provided \(\lambda^2 + 2kc^2 \geq 0\). \hfill \Box

3. Examples

In this section let’s see some examples of Schrödinger-Airy curves on Riemann surfaces.

**Example 3.1.** On the Euclidean plane, the round circles are Schrödinger-Airy curves for appropriate \(\lambda\).
Example 3.2. On the standard sphere $S^2$, except the great circle, any non-trivial intersection of a plane in $\mathbb{R}^3$ and the sphere $S^2$ gives rise to a Schrödinger-Airy curve.

For example, let $u : S^1 \to S^2$ be given by

$$u(x) = (\cos x \cos \phi, \sin x \cos \phi, \sin \phi), \quad x \in [0, 2\pi).$$

Here we suppose $\phi \in (0, \pi/2) \cup (\pi/2, \pi)$. That is, we exclude trivial maps and the geodesics. Then a direct computation yields:

$$u_x(x) = (-\sin x, \cos x, 0) \cos \phi,$$

$$\nabla_x u_x(x) = (-\cos x \sin \phi, -\sin x \sin \phi, \cos \phi) \sin \phi \cos \phi,$$

$$\nabla_x^2 u_x(x) = -(\sin^2 \phi)u_x.$$

Since the sphere has constant curvature 1, it follows that

$$R(u_x, Ju_x)Ju_x = |u_x|^2u_x = (\cos^2 \phi)u_x.$$

On the other hand, the complex structure on $S^2$ is given by $J(u) = u \times$. Thus

$$J(u)\nabla_x u_x = -(\sin x, -\cos x, 0) \sin \phi \cos \phi = -(\sin \phi)u_x.$$  

Therefore, $u$ is a Schrödinger-Airy curve with

$$\lambda = \frac{\sin^2 \phi - \frac{1}{2} \cos^2 \phi}{\sin \phi}.$$  

Particularly, if we choose $\phi = \arccos \sqrt{\frac{2}{3}}$, we get a non-trivial KdV curve.

Let us now turn our attention to the hyperbolic plane.

Example 3.3. Let $g_{-1} = dr^2 + (\sinh r)^2d\theta^2$ be the hyperbolic metric of constant curvature $-1$ on $\mathbb{R}^2$. We consider the curve $u : S^1 \to \mathbb{R}^2$ given by $u(x) = (r, x)$. It is clear that $|u_x| = \sinh r$ and

$$\nabla_x u_x = -(\sinh r)(\cosh r)\frac{\partial}{\partial r} = (\cosh r)Ju_x.$$  

Using this equation together with (2.2), we observe that

$$-\lambda(\cosh r)u_x = \lambda J\nabla_x u_x = \nabla_x^2 u_x + \frac{1}{2}R(u_x, Ju_x)Ju_x$$

$$= \nabla_x [(\cosh r)Ju_x] - \frac{1}{2}|u_x|^2u_x$$

$$= [-(\cosh r)^2 - \frac{1}{2}(\sinh r)^2]u_x.$$  

It follows that

$$\lambda = \frac{(\cosh r)^2 + \frac{1}{2}(\sinh r)^2}{\cosh r}.$$  

Note that $u$ has constant geodesic curvature $\cosh r / \sinh r$. Consequently, there are no non-trivial KdV curves on a hyperbolic plane.
4. Perturbed KdV curves on $T^{2n}$

By Theorem 1.2, the KdV curves on non-positively curved manifolds are trivial, so it is natural to consider the perturbed system. Namely, we shall consider the following system:

$$\begin{aligned}
\nabla_x^2 u_x + \frac{1}{2} R(u_x, J u_x) J u_x &= - J(u) \nabla_u H(x, u), \\
\end{aligned}$$

where $H(x, u)$ is a Hamiltonian function defined on $S^1 \times N$. Obviously, solutions to the above system are critical points of the perturbed functional

$$F_H(u) = F(u) + \int_{S^1} H(x, u) dx.$$ 

This is a strongly indefinite functional which has no lower bound. In general it is very hard to find critical points of $F_H$. Fortunately, in the special case of the torus, this problem can be significantly simplified.

In the following context, we suppose that $N = T^{2n}$ is the $2n$-dimensional flat torus and use $t$ to denote the variable in $S^1$ instead of $x$. Then the system (4.1) is reduced to

$$\frac{d^3 u}{dt^3} = - J \nabla H_t(u), \quad t \in S^1.$$ 

System (4.2) is similar to the usual Hamiltonian system

$$\frac{du}{dt} = - J \nabla H_t(u).$$

It is well known that the solution to (4.3) corresponds to the fixed point of symplectomorphisms on $T^{2n}$, and the number of solutions is related to the famous Arnold conjecture, which is solved by Conley and Zehnder [3]. The method they used relies on a saddle point reduction, which is due to Amann [1]. Namely, finding periodic solutions to the infinite dimensional system (4.3) shifts to a finite dimensional variational problem by a Lyapunov-Schmidt reduction, and the Morse theory can be applied.

The same saddle point reduction can be applied to our system (4.2). But here we follow a simplified version which appears in Chang’s book [2]. Our main result is the following:

**Theorem 4.1.** Suppose $H_t(u) = H(t, u) \in C^2(S^1 \times T^{2n}, \mathbb{R})$. Then the system (4.2) has at least $2n + 1$ solutions. If in addition all solutions are non-degenerate, then there are $2^{2n}$ solutions to (4.2).

Before proving the theorem, let us first observe some basic facts. The operator

$$A := J \frac{d^3}{dt^3}$$

is defined on a dense subspace $D(A)$ of the Hilbert space

$$L := L^2(S^1, T^{2n}).$$

Any function in $L$ can be viewed as a periodic function in $L^2(S^1, \mathbb{R}^{2n})$. By Fourier series theory, a function $u \in L^2(S^1, \mathbb{R}^{2n})$ can always be decomposed into

$$u(t) = \sum_{j=1}^{n} \sum_{m=-\infty}^{+\infty} c_{mj} e^{-imt} \phi_j,$$
where \( \{ \phi_1, \cdots, \phi_n \} \) is a basis of \( \mathbb{C}^n = \mathbb{R}^{2n} \) and \( c_{mj} \) satisfies
\[
\sum_{m=-\infty}^{+\infty} |c_{mj}|^2 < \infty, j = 1, \cdots, 2n.
\]

Since
\[
J \frac{d^3}{dt^3} (e^{-imt} \phi_j) = -m^3 e^{-imt} \phi_j,
\]
it is obvious that \( A \) is a selfadjoint operator which only has point spectrum \( \{ -m^3; m \in \mathbb{Z} \} \), and the eigenspace of the eigenvalue \( \lambda = -m^3 \) is
\[
(4.4) \quad \Lambda(-m^3) = \text{span}\{ e^{-imt} \phi_1, \cdots, e^{-imt} \phi_n \}.
\]

We shall work on the space \( V = D(\|A\|^\frac{1}{2}) \) defined by
\[
\sum_{j=1}^{n} \sum_{m=-\infty}^{+\infty} (1 + |m|^3) |c_{mj}|^2 < \infty.
\]

There is a well-defined functional on \( V \) given by
\[
(4.5) \quad a(v) = \frac{1}{2} \int_{S^1} \langle Av, v \rangle dt - \int_{S^1} H_t(v) dt, v \in V.
\]

Just as in the theory of the Hamiltonian system (4.3), it is easy to verify that the Euler-Lagrange equation of \( a \) is exactly the system (4.2). A solution \( v \) is called non-degenerate if the Hessian \( d^2 a(v) \) is an isomorphism.

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Since the solutions to equation (4.2) are just critical points of the functional \( a \in C^2(V, \mathbb{R}^1) \), we only have to investigate the number of critical points of \( a \). We prove this theorem in two steps. First, we show that the required critical points of \( a \) are in one-one correspondence with a function \( h \), which is defined on a finite dimensional space. Then we show that \( h \) satisfies the P.S. condition, so that the standard Morse theory yields the desired conclusion.

As previously noted, the selfadjoint operator \( A \) only has a discrete spectrum. So there exists \( \epsilon > 0 \) small, such that \( -\epsilon \) is not in the spectrum and \( A_\epsilon := \epsilon I + A \) is invertible.

On the other hand, since the Hamiltonian \( H(t, u) \in C^2(S^1 \times T^{2n}) \) is defined on a compact space, we may suppose that there exists a real number \( B > 0 \) such that
\[
\|\nabla_x H_t\| \leq B, \forall t \in S^1.
\]

The following step is to decompose the domain space \( V = D(\|A\|^\frac{1}{2}) \). First, for each eigenvalue \( \lambda \) of \( A \), let \( P(\lambda) \) be the projection from \( L \) to the eigenspace \( \Lambda(\lambda) \).
Let
\[
P_0 = \sum_{-B \leq \lambda \leq B} P(\lambda), P_+ = \sum_{\lambda < -B} P(\lambda), P_- = \sum_{\lambda > B} P(\lambda),
\]
and let \( L_0 = P_0 L, L_\pm = P_\pm L \). We also denote
\[
E_+ = \sum_{\lambda > 0} P(\lambda), E_- = \sum_{\lambda < 0} P(\lambda).
\]

Next, using the invertible operator \( A_\epsilon \), we may decompose the space \( V = D(\|A\|^\frac{1}{2}) \) into
\[
V = V_0 \oplus V_+ \oplus V_-,
\]
where $V_0 = |A_e|^{-\frac{1}{2}}L_0, V_{\pm} = |A_e|^{-\frac{1}{2}}L_{\pm}$. Define the graph norm on $V$ by
\[
\|v\|_V^2 := \|A^\frac{1}{2}v\|_L^2 + \epsilon^2\|v\|_L^2.
\]
Then for any $u \in L$ and $v = |A_e|^{-\frac{1}{2}}u \in V$, we have $\|u\|_L = \|v\|_V$. Moreover, we have the following decomposition:
\[
u = u_+ + u_- + u_0, \quad v = v_+ + v_- + v_0,
\]
where $u_0 = P_0u, u_{\pm} = P_{\pm}u$, and $v_0 = |A_e|^{-\frac{1}{2}}u_0, v_{\pm} = |A_e|^{-\frac{1}{2}}u_{\pm}$.

Now we define a functional on $L$ as follows:
\[
b(u) = \frac{1}{2}(\|u_+\|_L^2 + \|E_+u_0\|_L^2 - \|E_-u_0\|_L^2 - \|u_-\|_L^2) - \Phi_{\epsilon}(v),
\]
where
\[
\Phi_{\epsilon}(v) = \frac{\epsilon}{2}\|v\|_L^2 + \int_{S^1} H_{\epsilon}(v)dt.
\]
Obviously, this functional $b$ is actually the same as the functional $a$ given by (4.5) such that $a(v) = b(u)$. Besides, $u$ is a critical point of $b$ if and only if $v$ is a critical point of $a$. The Euler-Lagrange equation of $b$ for $u_{\pm}$ is
\[
u_{\pm} = \pm|A_e|^{-\frac{1}{2}}P_{\pm}F_{\epsilon}(v),
\]
where
\[
F_{\epsilon} = \epsilon I + \nabla H_{\epsilon} \in C^1(V, V).
\]
However, equation (4.6) is equivalent to
\[
v_{\pm} = A^{-1}_zP_{\pm}F_{\epsilon}(v_+ + v_- + v_0).
\]
A direct computation shows that the operator
\[
F_{\pm} := A^{-1}_zP_{\pm}F_{\epsilon} \in C^1(V, V)
\]
is a contraction. It follows by the implicit function theorem that there exists a solution $v_{\pm}(v_0)$ to equation (4.7) for fixed $v_0 \in V_0$. Thus we may define a functional on $V_0$ by
\[
h(z) = a(v(z)) = a(v_+(z) + v_-(z) + z), z \in V_0.
\]
One verifies readily that $z$ is a critical point of $h$ if and only if $v(z)$ is a critical point of $a$. As a consequence, the problem of finding critical points of $a$ is now reduced to finding critical points of a $C^2$ functional $h$ defined on a finite dimensional space $V_0$.

To proceed, we decompose $V_0$ into $V_0 = Z_\ast \oplus Z_0$, where $Z_\ast = E_+V_0 \oplus E_-V_0$, and $Z_0$ is the kernel of $A$. Let $z = z_\ast + z_0$, where $z_\ast \in Z_\ast, z_0 \in Z_0$. Because $H_{\epsilon}$ is periodic in $v$ and $(v_{\pm})$ is periodic in $z$ by equation (4.7), it follows that $h(z)$ is periodic in $z_0$. Note that $Z_0 = \Lambda(0)$ is a $2n$-dimensional space by (4.4). So $h(z)$ may be viewed as a functional defined on $Z_\ast \times T^{2n}$. If we let
\[
g(z) = \frac{1}{2}(A(v_+(z) + v_-(z)), v_+(z) + v_-(z)) - \int_{S^1} H_{\epsilon}(v(z))dt,
\]
then the functional $h$ is in the following form:
\[
h(z) = h(z_\ast, z_0) = \frac{1}{2}(Az_\ast, z_\ast) + g(z_\ast, z_0).
\]
It is easy to see that $dg(z) = P_0\nabla H_{\epsilon}(v(z))$ is bounded. Now by a standard argument in Morse theory (for example, see Theorem 5.3 in [2]), $f$ satisfies the P.S. condition.
and Theorem 4.1 is proved, provided the cuplength of $T^{2n}$ is $2n$ and the sum of the Betti numbers is $2^{2n}$.

\[ \square \]

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**References**


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